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Explicit resolutions for the complex of several Fueter operators

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Abstract

An analogue of the Dolbeault complex is introduced for regular functions of several quaternionic variables and studied by means of two different methods. The first one comes from algebraic analysis (for a thorough treatment see the book [F. Colombo, I. Sabadini, F. Sommen, D.C. Struppa, Analysis of Dirac systems and computational algebra, Progress in Mathematical Physics, Vol. 39, Birkhäuser, Boston, 2004]), while the other one relies on the symmetry of the equations and the methods of representation theory (see [F. Colombo, V. Souček, D.C. Struppa, Invariant resolutions for several Fueter operators, J. Geom. Phys. 56 (2006) 1175–1191; R.J. Baston, Quaternionic Complexes, J. Geom. Phys. 8 (1992) 29–52]). The comparison of the two results allows one to describe the operators appearing in the complex in an explicit form. This description leads to a duality theorem which is the generalization of the classical Martineau–Harvey theorem and which is related to hyperfunctions of several quaternionic variables.

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1. Introduction

In recent years, a lot of attention has been devoted to problems related to the generalization of the theory of several complex variables to higher dimension, i.e., to the study of the nullsolutions of several Dirac operators in a Clifford algebra setting. The purpose of most of these works is the study of an analogue of the Dolbeault sequence in which the first operator (in higher dimension) is given by several Dirac operators. The methods used came either from algebraic analysis, and were supported by computational tools like the theory of Gröbner bases (see [1,3,6]), or from Clifford analysis (see [10,11]). In the paper [7] the authors exploited another approach, based on symmetry considerations and on representation theory (see [4,12–14]).

We point out that in algebraic analysis, no attention is paid to the invariance properties of the operators involved. The standard procedure used to compute the complex is the explicit computation, step by step, of the syzygies of the

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maps appearing in the complex. The complexity of such a computation is doubly exponential, as was shown in a well known paper of Bayer and Stillman [5]. If the given operator has a known symmetry, we can use this information to reduce the computational complexity. In general, if the first operator in the sequence is invariant with respect to a certain symmetry, the same property is shared by all the other operators in the resolution. The symmetry, in the case of several Cauchy–Fueter operators, has been studied in [7] and this information will be used in this paper to show the explicit form of the maps in the complex. The explicit form of the last map in the complex leads to a duality theorem generalizing the classical Martineau–Harvey duality theorem to the quaternionic setting and thus describing analytic functionals (which are related to hyperfunctions).

2. Notation and preliminary results

The algebra of quaternions will be denoted by $\mathbb{H}$, while a quaternion will be written as $q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, where $x_\ell \in \mathbb{R}$ for $\ell = 0, \dots, 3$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the imaginary units. The algebra $\mathbb{H}$ will be identified with $\mathbb{C} + \mathbf{j}\mathbb{C}$ and we will write a quaternion as $q = u_1 + \mathbf{j}u_2$ where $u_1 = x_0 + \mathbf{i}x_1$ and $u_2 = x_2 - \mathbf{i}x_3$. The algebra $\mathbb{H}$ can also be represented by 2×2 matrices with complex entries. For $A = 0, 1$, $A' = 0', 1'$ we have

$$q \cong \eta_{AA'} = \begin{bmatrix} \eta_{00'} & \eta_{01'} \\ \eta_{10'} & \eta_{11'} \end{bmatrix} = \begin{bmatrix} x_0 + \mathrm{i}x_1 & -x_2 - \mathrm{i}x_3 \\ x_2 - \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{bmatrix}, \quad \mathrm{i} = \sqrt{-1}. \tag{1}$$

A natural generalization of the Cauchy–Riemann operator to this setting is the so called Cauchy–Fueter operator which is defined as (see [15])

$$\frac{\partial}{\partial \bar{q}} = \partial_{x_0} + \mathbf{i}\partial_{x_1} + \mathbf{j}\partial_{x_2} + \mathbf{k}\partial_{x_3},$$

with the obvious meaning of the symbols. The kernel of $\partial/\partial\bar{q}$ for differentiable functions gives the so called left regular functions. For the sequel, it is useful to express the regularity condition in real components: a function f is left regular if and only if its four real components f_0, f_1, f_2, f_3 satisfy the following 4×4 system of linear constant coefficient differential equations

$$\begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ \partial_{x_1} & \partial_{x_0} & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_2} & \partial_{x_3} & \partial_{x_0} & -\partial_{x_1} \\ \partial_{x_3} & -\partial_{x_2} & \partial_{x_1} & \partial_{x_0} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0.$$

To simplify the notation, we will write the previous condition as a matrix multiplication

$$U(D)\vec{f} = 0$$

and, when considering several quaternionic variables $q_\ell = x_{\ell 0} + ix_{\ell 1} + jx_{\ell 2} + kx_{\ell 3}$, we will write $U_\ell(D)\vec{f} = 0$.

Using the matrix notation (1), the Cauchy–Fueter operator becomes

$$\frac{\partial}{\partial \bar{q}} \cong \begin{bmatrix} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{bmatrix} = \begin{bmatrix} \partial_{x_0} + \mathrm{i}\partial_{x_1} & -\partial_{x_2} - \mathrm{i}\partial_{x_3} \\ \partial_{x_2} - \mathrm{i}\partial_{x_3} & \partial_{x_0} - \mathrm{i}\partial_{x_1} \end{bmatrix}, \tag{2}$$

while the regularity condition can be written, using the spinor notation, in the form

$$\begin{bmatrix} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{bmatrix} \begin{bmatrix} \varphi^{0'} \\ \varphi^{1'} \end{bmatrix} = 0 \tag{3}$$

where we have set $\varphi^{0'} := f_0 + \mathrm{i} f_1$ and $\varphi^{1'} := f_2 - \mathrm{i} f_3$. Setting

$$\nabla_{AA'} = \begin{bmatrix} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{bmatrix},$$

the two equations in (3) can be written as

$$\nabla_{AA'}\varphi^{A'} = 0, \quad A = 0, 1.$$

In the paper [7] the authors prove that the Cauchy–Fueter complex can be obtained either using an algebraic approach based on Gröbner bases techniques or using the theory of invariant operators. We quickly summarize their results for sake of completeness.

To get the complex of n Cauchy–Fueter operators in an algebraic way, we consider the system

$$\begin{bmatrix} U_1(D) \\ \dots \\ U_n(D) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = P(D)\vec{f} = 0.$$

The algebraic object which encodes some analytic information of the system is the module $M = \operatorname{coker} P^t$ where P is the $4n \times 4$ matrix symbol of $P(D)$. The matrix P has entries in the ring of polynomials $R = \mathbb{C}[x_{01}, x_{11}, x_{21}, x_{31}, \dots, x_{0n}, x_{1n}, x_{2n}, x_{3n}]$. A finite free resolution of the module M can always be constructed according to what is usually called Hilbert's syzygy theorem. The maps which appear in the resolution are called the syzygies of M, and they can be constructed in several different ways; thus such a resolution is not unique. Nonetheless, with a minimal choice of generators at each step, one obtains a *minimal* free resolution in which the ranks of the free modules, i.e. the so called Betti numbers, only depend on the module M and not on the choice of the syzygies. The resolution in the case of the module M associated with the Cauchy–Fueter system in $n > 1$ variables is very well known (see [2,3,6]) and it can be dualized through the use of the Hom functor to obtain:

$$0 \longrightarrow R^{r_0} \xrightarrow{P} R^{r_1} \xrightarrow{P_1} \cdots \longrightarrow R^{r_{2n-2}} \xrightarrow{P_{2n-2}} R^{r_{2n-1}} \longrightarrow 0. \tag{4}$$

The same complex can be obtained via the representation theory as described in detail in [7]. The result is the sequence

$$0 \longrightarrow \mathbb{C}^2 \xrightarrow{D_0} \mathbb{C}^{2n} \xrightarrow{D_1} \Lambda^3(\mathbb{C}^{2n}) \xrightarrow{D_2} \mathbb{C}^2 \otimes \Lambda^4(\mathbb{C}^{2n}) \longrightarrow \cdots \longrightarrow \odot^{2n-3}(\mathbb{C}^2) \otimes \Lambda^{2n}(\mathbb{C}^{2n}) \longrightarrow 0. \tag{5}$$

The operators $D_j,\ j = 0, 1, \dots, 2n-2$, are given by the composition of the invariant projection π with the gradient $\nabla_{\alpha A'}\varphi$, of the field φ (or with the second gradient $\nabla_{\beta B'}\nabla_{\alpha A'}\varphi$). We revise their description for sake of completeness. An element of the representation $\mathbb{C}^2$ will be denoted by $\varphi^{A'}$, $A' = 0, 1$, while elements in the symmetric power $\odot^j(\mathbb{C}^2)$ are symmetric tensor fields denoted by $\varphi^{A'\dots E'}$, with j capital roman indices. Finally, elements of the outer power $\Lambda^k(\mathbb{C}^{2n})$, $k = 1, \dots, 2n$, are antisymmetric tensor fields denoted by $\varphi_{\alpha,\dots,\gamma}$, with k Greek indices. The symbol $\nabla_{A'\alpha}$, $A' = 0, 1$, $\alpha = 1, \dots, 2n$, represents the gradient, as implicitly defined in (2).

The operator D_0 from functions with values in $\mathbb{C}^2$ to functions with values $\mathbb{C}^{2n}$ can be written as

$$[D_0(\varphi^{A'})]_\alpha = \nabla_{A'\alpha}\varphi^{A'}.$$

The operator D_1, of second order, is defined by

$$[D_1(\varphi_\gamma)]_{\alpha\beta\gamma} = \nabla_{A'[\alpha}\nabla_\beta^{A'}\varphi_{\gamma]}, \tag{6}$$

where the brackets $[\dots]$ mean total antisymmetrization in the corresponding indices. All the other operators D_j are of first order: D_j is defined on fields with $j-2$ upper indices and $j+1$ lower indices by

$$[D_j(\varphi_{\beta\dots\delta}^{B'\dots F'})]_{\alpha\dots\delta}^{A'\dots F'} = \nabla_{[\alpha}^{(A'}\varphi_{\beta\dots\delta]}^{B'\dots F')}, \tag{7}$$

where the round parentheses $(\dots)$ mean the symmetrization in the corresponding indices.

3. The complex for two operators

In this section we will explicitly show how the complex in two quaternionic variables can be treated with the two different approaches described in the previous section, and we show how to translate one description into the other.

Theorem 3.1. *The Cauchy–Fueter complex for two operators constructed through the Hilbert syzygy theorem coincides with the complex constructed through invariant operator theory.*

Proof. Let us consider two Cauchy–Fueter operators $\partial/\partial\bar{q}_i = \partial_{\bar{q}_i}$, $i = 1, 2$, and the corresponding system:

$$\begin{cases} \partial_{\bar{q}_1} f = g_1 \\ \partial_{\bar{q}_2} f = g_2. \end{cases}$$

This system can be translated into another system of eight real equations that can be written in matrix form (see Section 2) as

$$P(D)\vec{f} = 0.$$

By considering the Fourier transform of this matrix we get

$$P = \begin{bmatrix} x_{10} & -x_{11} & -x_{12} & -x_{13} \\ x_{11} & x_{01} & -x_{31} & x_{21} \\ x_{21} & x_{31} & x_{01} & -x_{11} \\ x_{31} & -x_{21} & x_{11} & x_{01} \\ x_{02} & -x_{12} & -x_{22} & -x_{32} \\ x_{12} & x_{02} & -x_{32} & x_{22} \\ x_{22} & x_{32} & x_{02} & -x_{12} \\ x_{23} & -x_{22} & x_{21} & x_{20} \end{bmatrix}$$

and the minimal free resolution of the module $M = \operatorname{coker} P^t$ is

$$0 \longrightarrow R^4 \xrightarrow{P_2^t} R^8 \xrightarrow{P_1^t} R^8 \xrightarrow{P^t} R^4 \longrightarrow M \longrightarrow 0$$

which translates into the complex of operators

$$0 \longrightarrow \mathcal{S}(U)^4 \xrightarrow{P(D)} \mathcal{S}(U)^8 \xrightarrow{P_1(D)} \mathcal{S}(U)^8 \xrightarrow{P_2(D)} \mathcal{S}(U)^4 \longrightarrow 0 \tag{8}$$

where $\mathcal{S}$ is, for example, the sheaf of $\mathcal{C}^\infty$ functions, although one can use other sheaves of generalized functions such as distributions or hyperfunctions. We know from the general theory of the Cauchy–Fueter complex (see [1–3, 6]) that the two quaternionic relations coming from the matrix $P_1(D)$ give the following quaternionic compatibility conditions:

$$\begin{cases} \partial_{\bar{q}_1}\partial_{q_1} g_2 - \partial_{\bar{q}_2}\partial_{q_1} g_1 = 0 \\ \partial_{\bar{q}_2}\partial_{q_2} g_1 - \partial_{\bar{q}_1}\partial_{q_2} g_2 = 0. \end{cases} \tag{9}$$

The complex closes with one more linear condition that is the compatibility condition for the solvability of the system

$$\begin{cases} \partial_{\bar{q}_1}\partial_{q_1} g_2 - \partial_{\bar{q}_2}\partial_{q_1} g_1 = h_{12} \\ \partial_{\bar{q}_2}\partial_{q_2} g_1 - \partial_{\bar{q}_1}\partial_{q_2} g_2 = h_{21}. \end{cases} \tag{10}$$

One can easily verify that the condition, coming from $P_2(D)$, is

$$\partial_{q_1} h_{21} + \partial_{q_2} h_{12} = 0. \tag{11}$$

Now we consider the description arising from invariant operator theory. We define the usual Cauchy–Fueter operators (and their conjugates) as

$$\partial_{\bar{q}_i} \cong \nabla^i_{AA'} = \begin{bmatrix} \nabla^i_{00'} & \nabla^i_{01'} \\ \nabla^i_{10'} & \nabla^i_{11'} \end{bmatrix}, \qquad \partial_{q_i} \cong \overline{\nabla}^i_{AA'} = \begin{bmatrix} \nabla^i_{11'} & -\nabla^i_{01'} \\ -\nabla^i_{10'} & \nabla^i_{00'} \end{bmatrix}. \tag{12}$$

Taking into account the above definitions we have that

$$\partial_{\bar{q}_i}\partial_{q_j} \cong \begin{bmatrix} \nabla^i_{00'}\nabla^j_{11'} - \nabla^j_{10'}\nabla^i_{01'} & -\nabla^i_{00'}\nabla^j_{01'} + \nabla^i_{01'}\nabla^j_{00'} \\ \nabla^i_{10'}\nabla^j_{11'} - \nabla^i_{11'}\nabla^j_{10'} & -\nabla^i_{10'}\nabla^j_{01'} + \nabla^i_{11'}\nabla^j_{00'} \end{bmatrix}.$$

In particular with $i = j$ we have the Laplace operators

$$\partial_{\bar{q}_i}\partial_{q_i} = \partial_{q_i}\partial_{\bar{q}_i} \cong \begin{bmatrix} \nabla^i_{00'}\nabla^i_{11'} - \nabla^i_{01'}\nabla^i_{10'} & 0 \\ 0 & \nabla^i_{11'}\nabla^i_{00'} - \nabla^i_{10'}\nabla^i_{01'} \end{bmatrix}.$$

According to the discussion in [7], the complex in the case of two operators can be described as follows:

$$0 \longrightarrow \mathbb{C}^2 \xrightarrow{D_0} \mathbb{C}^4 \xrightarrow{D_1} \Lambda^3(\mathbb{C}^4) \xrightarrow{D_2} \mathbb{C}^2 \otimes \Lambda^4(\mathbb{C}^4) \longrightarrow 0.$$

The compatibility relations on the data of the non-homogeneous Cauchy–Fueter system

$$\nabla^i_{A\,[A'}\varphi_{B']} = \psi^i_A, \quad i = 1, 2,\ A, B \in \{0, 1\},$$

can be constructed via antisymmetrization according to (6). It is possible to show that the compatibility conditions are given by the 3×3 minors of the matrix

$$\begin{bmatrix} \nabla^1_{00'} & \nabla^1_{01'} & \psi^1_0 \\ \nabla^1_{10'} & \nabla^1_{11'} & \psi^1_1 \\ \nabla^2_{00'} & \nabla^2_{01'} & \psi^2_0 \\ \nabla^2_{10'} & \nabla^2_{11'} & \psi^2_1 \end{bmatrix}.$$

In fact, consider the four different minors

$$M^{ij}_A = \begin{vmatrix} \nabla^i_{00'} & \nabla^i_{01'} & \psi^i_0 \\ \nabla^i_{10'} & \nabla^i_{11'} & \psi^i_1 \\ \nabla^j_{A0'} & \nabla^j_{A1'} & \psi^j_A \end{vmatrix} \quad i, j \in \{1, 2\},\ A \in \{0, 1\}$$

and observe that the relations

$$M^{12}_0 = 0, \qquad M^{12}_1 = 0$$

can be written as

$$\begin{bmatrix} M^{12}_0 \\ M^{12}_1 \end{bmatrix} = \begin{bmatrix} \nabla^1_{00'}\nabla^2_{11'} - \nabla^1_{01'}\nabla^2_{10'} & -\nabla^1_{00'}\nabla^2_{10'} + \nabla^1_{01'}\nabla^2_{00'} & \Delta_1 & 0 \\ \nabla^1_{10'}\nabla^2_{11'} - \nabla^1_{11'}\nabla^2_{10'} & -\nabla^1_{10'}\nabla^2_{01'} + \nabla^1_{11'}\nabla^2_{00'} & 0 & \Delta_1 \end{bmatrix} \begin{bmatrix} \psi^1_0 \\ \psi^1_1 \\ \psi^2_0 \\ \psi^2_1 \end{bmatrix} = 0. \tag{13}$$

By setting $g_i = [\psi^i_0\ \psi^i_1]^t$, one verifies that condition (13) corresponds to

$$\begin{bmatrix} -\partial_{\bar q_2}\partial_{q_1} & \partial_{\bar q_1}\partial_{q_1} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = 0,$$

and, analogously, the relations

$$M^{21}_0 = 0, \qquad M^{21}_1 = 0 \tag{14}$$

correspond to $\partial_{\bar q_2}\partial_{q_2} g_1 - \partial_{\bar q_1}\partial_{q_2} g_2 = 0$.

Let us consider the inhomogeneous system arising from (13) and (14):

$$\begin{cases} M^{12}_0 = \phi^1_0 \\ M^{12}_1 = \phi^1_1 \\ M^{21}_0 = \phi^2_0 \\ M^{21}_1 = \phi^2_1. \end{cases}$$

To close the complex we need to consider the relations one gets taking the suitable symmetrization and antisymmetrization of the indices. Once again, it is equivalent to consider the determinants of the two 4×4 matrices

$$\begin{bmatrix} \nabla^1_{0A'} & \nabla^1_{00'} & \nabla^1_{01'} & \psi^1_0 \\ \nabla^1_{1A'} & \nabla^1_{10'} & \nabla^1_{11'} & \psi^1_1 \\ \nabla^2_{0A'} & \nabla^2_{00'} & \nabla^2_{01'} & \psi^2_0 \\ \nabla^2_{1A'} & \nabla^2_{10'} & \nabla^2_{11'} & \psi^2_1 \end{bmatrix} \quad A \in \{0, 1\}.$$

The determinants, for $A = 0, 1$, give the two conditions:

$$\nabla^1_{00'}\phi^2_1 - \nabla^1_{10'}\phi^2_0 + \nabla^2_{00'}\phi^1_1 - \nabla^2_{10'}\phi^1_0 = 0,$$
$$\nabla^1_{01'}\phi^2_1 - \nabla^1_{11'}\phi^2_0 + \nabla^2_{01'}\phi^1_1 - \nabla^2_{11'}\phi^1_0 = 0,$$

that, in matrix form, can be written in the form

$$\begin{bmatrix} \nabla^2_{11'} & -\nabla^2_{01'} & \nabla^1_{11'} & -\nabla^1_{01'} \\ -\nabla^2_{10'} & \nabla^2_{00'} & -\nabla^1_{10'} & \nabla^1_{00'} \end{bmatrix} \begin{bmatrix} \phi^1_0 \\ \phi^1_1 \\ \phi^2_0 \\ \phi^2_1 \end{bmatrix} = 0. \tag{15}$$

Using (12) and setting $h_{12} = [\phi^1_0\ \phi^1_1]$, $h_{21} = [\phi^2_0\ \phi^2_1]$, it is immediately verified that (15) corresponds to the relation (11) in the complex (8). □

Remark 3.2. The description of the maps in the complex of two Cauchy–Fueter operators is not new; see e.g. [6]. In the case of the algebraic construction, the fact that the relations found are not only necessary but also sufficient was proved with the use of CoCoA which provides the minimal number of relations at each step. However CoCoA (and similar computer algebra packages) cannot display the relations in quaternionic form since the syzygies are written in real components and, in general, it is not possible to automatically group the various real relations to obtain quaternionic ones. The main advantage of the construction through the representation theory is that it provides a method for writing the relations in the complex explicitly, taking also into account the invariance of the operators involved, so that they can be more easily grouped to produce quaternionic relations.

We conclude this section by presenting a duality theorem which is related to the definition of hyperfunctions in two quaternionic variables through the last map $P_2(D)$ of the Cauchy–Fueter complex (see [6], Theorem 2.1.11 and [9]). To state the theorem and its proof, we set the following definitions. Let $\mathcal{S}$ be the sheaf of infinitely differentiable functions, let $\mathcal{R}$ (resp. $\tilde{\mathcal{R}}$) be the sheaf of left regular (resp. anti-regular) functions in $\mathbb{H}^2$, and let $\mathcal{S}^Q$ be the sheaf of infinitely differentiable functions solutions to the equation $Q(D)f = 0$. Then we have:

Theorem 3.3. *Let K be a compact convex set in $\mathbb{H}^2$. Let $P_2(D)$ be as in (8) and let $Q(D) = P^t_2(D)$. Then we have:*

$$H^3_K(\mathbb{H}^2, \tilde{\mathcal{R}}) \cong [\mathcal{R}(K)]'. \tag{16}$$

Proof. It is known (see [1]) that $\mathrm{Ext}^j_R(M, R) = 0$ for $j = 0, 1, 2$, so we have

$$H^3_K(\mathbb{H}^2, \mathcal{S}^Q) \cong [\mathcal{R}(K)]'.$$

The proof of Theorem 4.1 shows that the matrix $P_2(D)$ is associated with the operator $[\partial/\partial q_2\ \ \partial/\partial q_1]$ and so $P_2(D) = [U^t_2(D)\quad U^t_1(D)]$. Since $Q(D) = P^t_2(D)$ we get the statement. □

4. The complex for $n \geq 3$ operators

We now consider $n \geq 3$ operators. In this case we know the length of the complex, the number of relations at each step and their degree; see [3,6,7]. The explicit description of the first syzygies is known, but unknown, so far, has been an explicit description of the other maps appearing in the complex. The presence of the exceptional syzygies (see below) which involves operators containing only two of the four possible derivatives makes it hard (and perhaps impossible) to write the next relations using only the Cauchy–Fueter operators in the various quaternionic variables. The procedure we will illustrate in this paper allows us to provide the needed description of all the maps. The first syzygies appearing in the complex are as in the next result, in which we show that they can be equivalently obtained using the description (5).

Theorem 4.1. *The compatibility conditions of the system*

$$\begin{cases} \partial_{\bar{q}_1} & = g_1 \\ \dots & \dots \\ \partial_{\bar{q}_n} & = g_n \end{cases}$$

are the following:

(1) *for each of the* $2\binom{n}{2}$ *ordered pairs of indices* r, s, $1 \le r, s \le n$,

$$\partial_{\bar{q}_r}\partial_{q_s} g_s - \partial_{\bar{q}_s}\partial_{q_s} g_r = 0$$

(2) *for each of the* $\binom{n}{3}$ *triples of indices* h, r, s, $\quad 1 \le h, r, s \le n$,

$$\partial_{q_h}\partial_{\bar{q}_r} g_s + \partial_{q_r}\partial_{\bar{q}_h} g_s - \partial_{\bar{q}_s}\partial_{q_r} g_h - \partial_{\bar{q}_s}\partial_{q_h} g_r = 0$$

and

$$\partial_{q_r}\partial_{\bar{q}_s} g_h + \partial_{q_s}\partial_{\bar{q}_r} g_h - \partial_{\bar{q}_h}\partial_{q_r} g_s - \partial_{\bar{q}_h}\partial_{q_s} g_r = 0,$$

(3) *for each of the* $\binom{n}{3}$ *triples of indices* h, r, s, $\quad 1 \le h, r, s \le n$,

$$(D_{q_r}\partial_{\bar{q}_s} - D_{q_s}\partial_{\bar{q}_r})g_h + (D_{q_s}\partial_{\bar{q}_h} - D_{q_h}\partial_{\bar{q}_s})g_r + (D_{q_h}\partial_{\bar{q}_r} - D_{q_r}\partial_{\bar{q}_h})g_s = 0,$$

$$(D'_{q_r}\partial_{\bar{q}_s} - D'_{q_s}\partial_{\bar{q}_r})g_h + (D'_{q_s}\partial_{\bar{q}_h} - D'_{q_h}\partial_{\bar{q}_s})g_r + (D'_{q_h}\partial_{\bar{q}_r} - D'_{q_r}\partial_{\bar{q}_h})g_s = 0,$$

where

$$D_{q_i} = -\mathbf{j}\partial_{x_{i2}} + \mathbf{k}\partial_{x_{i3}}, \qquad D'_{q_i} = -\mathbf{i}\partial_{x_{i1}} + \mathbf{k}\partial_{x_{i3}}.$$

These conditions can be obtained via the complex (5).

Remark 4.2. Note that it is possible to write at least two other possible syzygies of the above form (3), but they are redundant:

$$\partial_{q_s}\partial_{\bar{q}_h} g_r + \partial_{q_h}\partial_{\bar{q}_s} g_r - \partial_{\bar{q}_r}\partial_{q_s} g_h - \partial_{\bar{q}_r}\partial_{q_h} g_s = 0$$

and

$$(D''_{q_r}\partial_{\bar{q}_s} - D''_{q_s}\partial_{\bar{q}_r})g_h + (D''_{q_s}\partial_{\bar{q}_h} - D''_{q_h}\partial_{\bar{q}_s})g_r + (D''_{q_h}\partial_{\bar{q}_r} - D''_{q_r}\partial_{\bar{q}_h})g_s = 0,$$

where

$$D''_{q_h} = \mathbf{i}\partial_{x_{i1}} - \mathbf{j}\partial_{x_{i2}}.$$

Proof. Let us consider first the case $n = 3$. The complex arising from this construction is

$$0 \longrightarrow \mathbb{C}^2 \xrightarrow{D_0} \mathbb{C}^6 \xrightarrow{D_1} \Lambda^3(\mathbb{C}^6) \xrightarrow{D_2} \mathbb{C}^2 \otimes \Lambda^4(\mathbb{C}^6) \xrightarrow{D_3} \odot^2(\mathbb{C}^2) \otimes \Lambda^5(\mathbb{C}^6) \xrightarrow{D_4} \odot^3(\mathbb{C}^2) \otimes \Lambda^6(\mathbb{C}^6) \longrightarrow 0. \quad (17)$$

From the previous discussion, one may argue that the 3×3 minors of the matrix

$$\begin{bmatrix} \nabla^1_{00'} & \nabla^1_{01'} & \psi^1_0 \\ \nabla^1_{10'} & \nabla^1_{11'} & \psi^1_1 \\ \nabla^2_{00'} & \nabla^2_{01'} & \psi^2_0 \\ \nabla^2_{10'} & \nabla^2_{11'} & \psi^2_1 \\ \nabla^3_{00'} & \nabla^3_{01'} & \psi^3_0 \\ \nabla^3_{10'} & \nabla^3_{11'} & \psi^3_1 \end{bmatrix} \quad (18)$$

give the compatibility relations on the data of the system

$$\nabla^i_{A[A'}\varphi_{B']} = \psi^i_A, \quad i = 1, 2, 3, \; A, B \in \{0, 1\}.$$

We define the following 12 minors for $A = 0, 1$; $i, j = 1, 2, 3$:

$$M_A^{ij} = \begin{vmatrix} \nabla^i_{00'} & \nabla^i_{01'} & \psi^i_0 \\ \nabla^i_{10'} & \nabla^i_{11'} & \psi^i_1 \\ \nabla^j_{A0'} & \nabla^j_{A1'} & \psi^j_A \end{vmatrix}$$

and also the 8 minors for $A, B, C = 0, 1$:

$$M_{ABC} = \begin{vmatrix} \nabla^1_{A0'} & \nabla^1_{A1'} & \psi^1_A \\ \nabla^2_{B0'} & \nabla^2_{B1'} & \psi^2_B \\ \nabla^3_{C0'} & \nabla^3_{C1'} & \psi^3_C \end{vmatrix}.$$

It is now possible to show that the minors M_A^{ij} correspond to the six quaternionic syzygies involving just two indices $i, j \in \{1, 2, 3\}$, i.e.

$$\partial_{\bar{q}_i}\partial_{q_i} g_j - \partial_{\bar{q}_j}\partial_{q_i} g_i = 0. \tag{19}$$

For sake of simplicity we work with $i = 1$ and $j = 2$, since the other cases are similar. The relation $M_A^{12} = 0$, $A = 0, 1$, can be written as

$$\begin{bmatrix} \nabla^1_{00'}\nabla^2_{11'} - \nabla^1_{01'}\nabla^2_{10'} & -\nabla^1_{00'}\nabla^2_{10'} + \nabla^1_{10'}\nabla^2_{00'} & \Delta_1 & 0 & 0 & 0 \\ \nabla^1_{01'}\nabla^2_{11'} - \nabla^1_{11'}\nabla^2_{01'} & -\nabla^1_{01'}\nabla^2_{10'} + \nabla^1_{11'}\nabla^2_{00'} & 0 & \Delta_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi^1_0 \\ \psi^1_1 \\ \psi^2_0 \\ \psi^2_1 \\ \psi^3_0 \\ \psi^3_1 \end{bmatrix} = 0 \tag{20}$$

where we have set $\Delta_1 = \nabla^1_{00'}\nabla^1_{11'} - \nabla^1_{01'}\nabla^1_{10'}$. We now observe that Eq. (20) corresponds to (19) for $i = 1$, $j = 2$. Similarly, the other syzygies involving only two indices i, j can be obtained as

$$M_0^{ij} = M_1^{ij} = 0.$$

Let us consider the syzygies of the form (2) in the statement. The triples of indices (h, r, s) take the values $1 \leq h, r, s \leq 3$ so that we get two independent relations, e.g.

$$\partial_{q_1}\partial_{\bar{q}_2} g_3 + \partial_{q_2}\partial_{\bar{q}_1} g_3 - \partial_{\bar{q}_3}\partial_{q_1} g_2 - \partial_{\bar{q}_3}\partial_{q_2} g_1 = 0 \tag{21}$$

and

$$\partial_{q_1}\partial_{\bar{q}_3} g_2 + \partial_{q_3}\partial_{\bar{q}_1} g_2 - \partial_{\bar{q}_2}\partial_{q_3} g_1 - \partial_{\bar{q}_2}\partial_{q_3} g_1 = 0. \tag{22}$$

We may verify that the syzygies above correspond to the following systems:

$$\begin{cases} M_{010} - M_{100} = 0 \\ M_{011} - M_{101} = 0 \end{cases} \quad \text{and} \quad \begin{cases} M_{110} - M_{011} = 0 \\ M_{100} - M_{001} = 0 \end{cases} \tag{23}$$

and, in fact, the system on the left in (23) can be written in matrix form as

$$\begin{bmatrix} -\nabla^3_{AA'}\bar{\nabla}^2_{AA'} & -\nabla^3_{AA'}\bar{\nabla}^1_{AA'} & \bar{\nabla}^1_{AA'}\nabla^2_{AA'} + \bar{\nabla}^2_{AA'}\nabla^1_{AA'} \end{bmatrix} \begin{bmatrix} \psi^1_0 \\ \psi^1_1 \\ \psi^2_0 \\ \psi^2_1 \\ \psi^3_0 \\ \psi^3_1 \end{bmatrix} = 0,$$

where $\nabla^i_{AA'}$, $\bar{\nabla}^i_{AA'}$, $i = 1, 2, 3$, are as in (12). By setting $g_i = [\psi^i_0\ \psi^i_1]^t$, it is easy to verify that this matrix relation corresponds to (21). In an analogous way, the system on the right corresponds to (22).

Let us consider the syzygies of the third type. The operators D_{q_i}, D'_{q_i}, D''_{q_i} can be represented by the following 2×2 matrices with complex entries:

$$D_{q_i} \cong \begin{bmatrix} 0 & \nabla^i_{01'} \\ \nabla^i_{10'} & 0 \end{bmatrix} \qquad D'_{q_i} \cong \frac{1}{2}\begin{bmatrix} \nabla^i_{11'} - \nabla^i_{00'} & \nabla^i_{10'} + \nabla^i_{01'} \\ \nabla^i_{01'} + \nabla^i_{10'} & \nabla^i_{00'} - \nabla^i_{11'} \end{bmatrix}$$

$$D''_{q_i} \cong \frac{1}{2}\begin{bmatrix} \nabla^i_{00'} - \nabla^i_{11'} & \nabla^i_{10'} - \nabla^i_{01'} \\ \nabla^i_{01'} - \nabla^i_{10'} & \nabla^i_{11'} - \nabla^i_{00'} \end{bmatrix}.$$

Consider:

$$(D_{q_2}\partial_{\bar{q}_3} - D_{q_3}\partial_{\bar{q}_2})g_1 + (D_{q_3}\partial_{\bar{q}_1} - D_{q_1}\partial_{\bar{q}_3})g_2 + (D_{q_1}\partial_{\bar{q}_2} - D_{q_2}\partial_{\bar{q}_1})g_3 = 0, \tag{24}$$

$$(D'_{q_2}\partial_{\bar{q}_3} - D'_{q_3}\partial_{\bar{q}_2})g_1 + (D'_{q_3}\partial_{\bar{q}_1} - D'_{q_1}\partial_{\bar{q}_3})g_2 + (D'_{q_1}\partial_{\bar{q}_2} - D'_{q_2}\partial_{\bar{q}_1})g_3 = 0. \tag{25}$$

With some computations similar to those already done in the other cases, one gets that the syzygies above correspond to the systems:

$$\begin{cases} M_{101} - M_{110} = 0 \\ M_{001} - M_{010} = 0 \end{cases} \quad \text{and} \quad \begin{cases} M_{000} = 0 \\ M_{111} = 0. \end{cases} \tag{26}$$

Note that this description is not affected by the fact we are considering only three variables, since it can be repeated for any choice of indices A, B, C in the definition of the matrices M^{ij}_A and M_{ABC}. So the proof holds in the general case $n \geq 3$. □

We can go further with the description of the maps in the complex. Let us consider again the case of three operators. We consider the non-homogeneous system

$$\begin{cases} \partial_{\bar{q}_r}\partial_{q_s}g_s - \partial_{\bar{q}_s}\partial_{q_s}g_r = h_{rs} \quad r, s = 1, 2, 3 \\ \partial_{q_1}\partial_{\bar{q}_2}g_3 + \partial_{q_2}\partial_{\bar{q}_1}g_3 - \partial_{\bar{q}_3}\partial_{q_2}g_1 - \partial_{\bar{q}_3}\partial_{q_1}g_2 = a_1 \\ \partial_{q_3}\partial_{\bar{q}_1}g_2 + \partial_{q_1}\partial_{\bar{q}_3}g_2 - \partial_{\bar{q}_2}\partial_{q_1}g_3 - \partial_{\bar{q}_2}\partial_{q_3}g_1 = a_2 \\ (D_{q_1}\partial_{\bar{q}_2} - D_{q_2}\partial_{\bar{q}_1})g_3 + (D_{q_2}\partial_{\bar{q}_3} - D_{q_3}\partial_{\bar{q}_2})g_1 + (D_{q_3}\partial_{\bar{q}_1} - D_{q_1}\partial_{\bar{q}_3})g_2 = b_1, \\ (D'_{q_1}\partial_{\bar{q}_2} - D'_{q_2}\partial_{\bar{q}_1})g_3 + (D'_{q_2}\partial_{\bar{q}_3} - D'_{q_3}\partial_{\bar{q}_2})g_1 + (D'_{q_3}\partial_{\bar{q}_1} - D'_{q_1}\partial_{\bar{q}_3})g_2 = b_2. \end{cases}$$

The second syzygies can be obtained as the maximal minors of the matrix obtained by adding to (18) a column of the type

$$\begin{bmatrix} \nabla^1_{0A'} \\ \nabla^1_{1A'} \\ \nabla^2_{0A'} \\ \nabla^2_{1A'} \\ \nabla^3_{0A'} \\ \nabla^3_{1A'} \end{bmatrix} \tag{27}$$

where $A = 0, 1$. This amounts to computing $\nabla^A_{[\alpha}M_{\beta\gamma\delta]}$ where $A = 0, 1$ and $\alpha, \ldots, \delta$ are different indices associated with the rows β, γ, δ of the matrix and thus varying in $\{1, \ldots, 6\}$. When dealing with 4×4 minors involving only two different upper indices, we get relations of the type (15) which can be rewritten as

$$\partial_{q_i}h_{ji} + \partial_{q_j}h_{ij} = 0.$$

Remark 4.3. We do not intend to write explicitly all the second syzygies, but we wish to point out that not all the relations can be written using the Cauchy–Fueter operator. For example, again in the case $n = 3$, we obtain

$$D_{q_3}h_{21} - D_{q_2}h_{31} - \frac{1}{3}D_{q_1}a_1 + \frac{1}{3}D_{q_1}a_2 - \check{D}_{q_1}b_1 - \frac{1}{3}\check{D}_{q_1}b_2 = 0$$

$$\check{D}_{q_3} h_{21} - \check{D}_{q_2} h_{31} - \frac{1}{3}\check{D}_{q_1} a_1 + \frac{1}{3}\check{D}_{q_1} a_2 - \check{D}_{q_1} b_1 - \frac{1}{3}\check{D}_{q_1} b_2 = 0,$$

where we have set $\check{D}_{q_i} = (\partial_{x_{0i}} + \mathbf{i}\partial_{x_{1i}})$.

Following the same procedure, we can write not only the third syzygies by computing the minors of the 5×6 matrices that we obtain by adding two columns of the type (27) to the matrix (18), but also all the other syzygies in the resolution. In the following theorem we describe the map $P^t_{2n-2}(D)$, the last in the complex.

Theorem 4.4. *The last map $P_{2n-2}(D)$ in the Cauchy–Fueter complex in $n \geq 3$ operators is associated with the operator:*

$$\begin{bmatrix} \partial_{q_1} & .. & \partial_{q_n} & 0 & .. & 0 & 0 & .. & 0 & .. & 0 & .. & 0 & 0 & .. & 0 \\ 0 & .. & 0 & \check{D}_{q_1} & .. & \check{D}_{q_n} & -D_{q_1} & .. & -D_{q_n} & .. & 0 & .. & 0 & 0 & .. & 0 \\ \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & .. & \vdots & .. & \vdots & \vdots & .. & \vdots \\ 0 & .. & 0 & 0 & .. & 0 & 0 & .. & 0 & .. & \check{D}_{q_1} & .. & \check{D}_{q_n} & -D_{q_1} & .. & -D_{q_n} \end{bmatrix}. \tag{28}$$

Proof. At the final step we have to consider the matrices obtained from

$$\begin{bmatrix} \nabla^1_{00'} & \nabla^1_{01'} & \psi^1_0 \\ \nabla^1_{10'} & \nabla^1_{11'} & \psi^1_1 \\ \nabla^2_{00'} & \nabla^2_{01'} & \psi^2_0 \\ \nabla^2_{10'} & \nabla^2_{11'} & \psi^2_1 \\ \vdots & \vdots & \vdots \\ \nabla^n_{00'} & \nabla^n_{01'} & \psi^n_0 \\ \nabla^n_{10'} & \nabla^n_{11'} & \psi^n_1 \end{bmatrix} \tag{29}$$

by adding $2n - 3$ columns of the type

$$\begin{bmatrix} \nabla^1_{0A'} \\ \nabla^1_{1A'} \\ \vdots \\ \nabla^n_{0A'} \\ \nabla^n_{1A'} \end{bmatrix}$$

where $A = 0, 1$. The index A runs on the $2n-2$ possibilities: $(0, 0, \dots, 0)$, $(1, 0, \dots, 0)$, $(1, 1, \dots, 0)$, $(1, 1, \dots, 1)$. We obtain $2n - 2$ square matrices of dimension $2n$ whose determinants can be written, according to the Laplace theorem, by multiplying each of the elements in the leftmost column by the corresponding minors. Note that only the first two possibilities $(0, 0, \dots, 0)$, $(1, 0, \dots, 0)$ involve the same $(2n - 1) \times (2n - 1)$ minors. Therefore, the last matrix can be written as

$$\begin{bmatrix} \nabla^1_{11'} & .. & -\nabla^n_{01'} & .. & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 & 0 \\ -\nabla^1_{10'} & .. & \nabla^n_{00'} & .. & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 & 0 \\ \vdots & .. & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & \vdots & .. & \vdots & \vdots \\ 0 & .. & 0 & .. & \nabla^1_{00'} & 0 & .. & \nabla^n_{00'} & 0 & 0 & -\nabla^1_{10'} & .. & 0 & -\nabla^1_{10'} \\ 0 & .. & 0 & .. & 0 & \nabla^1_{11'} & .. & 0 & \nabla^n_{11'} & -\nabla^1_{01'} & 0 & .. & -\nabla^n_{01'} & 0 \end{bmatrix} \tag{30}$$

or, in quaternionic form, as (28). □

As we have already pointed out, the last map $Q(D) = P^t_{2n-2}(D)$ in the complex is the most important in our description since it allows us to prove a duality theorem generalizing the classical Martineau–Harvey theorem (see e.g. [8]) which is related to the definition of hyperfunctions in several quaternionic variables (cf. Theorem 3.3). Then the sheaf $\mathcal{S}^Q$ of infinitely differentiable solutions to the equation $Q(D)F = 0$ can be described as follows:

Proposition 4.5. *The elements of the sheaf $\mathcal{S}^Q$ are $(n-1)$-tuples $F = (f_1, \ldots, f_{n-1})^t$ of infinitely differentiable functions such that f_j, $j = 1, \ldots, n-1$, are anti-regular with respect to the variables $q_1, \ldots, q_n$, and where f_j, $j \geq 2$, satisfy $(\partial_{x_{0\ell}} - \mathbf{j}\partial_{x_{2\ell}}) f_j = 0$ for any $\ell = 1, \ldots, n$.*

Proof. Let us consider the system $Q(D)F = 0$ where $Q(D) = P^t_{2n-2}(D)$ is given in (28). This translates into the fact that f_j, $j = 1, \ldots, n$, is anti-regular with respect to all the variables $q_1, \ldots, q_n$ while each f_j, $j = 2, \ldots n-1$, satisfies the system $(\partial_{x_{0\ell}} + \mathbf{i}\partial_{x_{1\ell}}) f_j = (\mathbf{j}\partial_{x_{2\ell}} - \mathbf{k}\partial_{x_{3\ell}}) f_j = 0$, for any $\ell = 1, \ldots n$. By taking into account anti-regularity and these last two relations, we get the statement. □

The duality theorem, whose proof is immediate if one observes that $\mathrm{Ext}^j_R(M, R) = 0$ for $j = 1, \ldots, 2n-2$ (see [6], Theorem 2.1.11 and [9]), is:

Theorem 4.6. *Let K be a compact convex set in $\mathbb{H}^n$ and set $Q = P^t_{2n-2}$. Let $\mathcal{S}^Q$ be as in Proposition 4.5 and let $\mathcal{R}$ be the sheaf of (left) regular functions. Then*

$$H^{2n-1}_K(\mathbb{H}^n, \mathcal{S}^Q) \cong [\mathcal{R}(K)]' \tag{31}$$

and

$$H^{2n-1}_K(\mathbb{H}^n, \mathcal{R}) \cong [\mathcal{S}^Q(K)]'.$$

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